The Cohomology of the Dyer Lashof Algebra

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The Dyer Lashof algebra R_p is a noncommutative algebra over Z/p similar in form to the mod p Steenrod algebra A_p. In this paper we show that the cohomology of the algebra R_p is isomorphic as A_p modules to $A_p(L)$, the Steenrod algebra for restricted Lie algebras. Many of the results of this paper are implicit in work of S. Priddy and of H. Miller for p = 2.

§ 1. Let X be an infinite loop space and let p be a prime. Then X has a product $m : X \times X \to X$ which enjoys strong homotopy commutativity properties. Kudo and Araki [KA] and Browder [B2] exploited these properties to construct mod 2 homology operations similar in type to the Steenrod reduced squares. Dyer and Lashof [DL] generalized this construction to odd primes and, in addition, showed that the operations $Q^k : H_n(X; Z/p) \to H_{n+2k(p-1)}(X; Z/p)$ satisfy excess and Adem relations quite analogous to those satisfied by the Steenrod operations.

Since $H^*(X; Z/p)$ is a module over A_p, $H_*(X; Z/p)$ is a module over A_p^{op}, the opposite Steenrod algebra. Thus $H_*(X; Z/p)$ is a module over both A_p^{op} and the algebra R_p of Dyer Lashof operations. Nishida [N] derived a formula for computing the interaction of these operations. J. P. May [CLM], [M3], [M4] has made extensive investigations and computations with this algebraic structure. See also Bisson [B1] and Madsen [M2].

DEFINITION 1.1. The Dyer Lashof algebra R_p is the graded Z/p algebra generated by symbols Q^k for $k \geq 0$ and βQ^k for $k > 0$ of degree $2k(p-1)$ and $2k(p-1)-1$ respectively. The relations are generated by the Adem relations

$$\beta^{\varepsilon} Q^r Q^s = \sum_j (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r} \beta^{\varepsilon} Q^{r+s-j} Q^j \qquad (1.2)$$

for $r > ps$

and

$$\beta^{\varepsilon} Q^r \beta Q^s = \sum_j (-1)^{r+j} \binom{(p-1)(j-s)}{pj-r} \beta^{\varepsilon} \beta Q^{r+s-j} Q^j$$

$$- \sum_j (-1)^{r+j} \binom{(p-1)(j-s)-1}{pj-r-1} \beta^{\varepsilon} Q^{r+s-j} \beta Q^j$$

for $r \geq ps$

where $\varepsilon = 0$ or 1 and $\beta\beta = 0$ where β is the mod p Bockstein.

REMARK 1.3. If $p = 2$ the Dyer Lashof algebra is usually defined as consisting of operations Q^k for $k \geq 0$ of degree k subject to the Adem relations above that do not contain β. It is easy to check that the mod 2 operation $\beta^{\varepsilon}Q^k$ in Definition 1.1 of degree $2k-\varepsilon$ satisfies the usual mod 2 relations for $Q^{2k-\varepsilon}$. Thus we need not make special cases in our theorems for p odd and even. In some examples below we will, however, use the standard mod 2 notation for Dyer Lashof operations.

Let A be a graded augmented Z/p algebra with augmentation ideal $\bar{A}$. The cohomology $H^{*,k}(A)=\mathrm{Ext}_A^{*,k}(Z/p, Z/p)$ is defined to be the cohomology of a projective resolution of the A module Z/p. For example, let $\bar{B}_{*,k}A = \bar{A} \otimes \ldots \otimes \bar{A}$ (k times) be the reduced bar construction on A. There are maps

$$d_j : \bar{B}_{*,k}A \to \bar{B}_{*,k-1}A \text{ for } j = 1, \ldots, k-1$$

given by $d_j(a_1 \otimes \ldots \otimes a_k) = a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_k$.

It is well known that, with $d = \sum (-1)^j d_j : \bar{B}_{*,k}A \to \bar{B}_{*,k-1}A$, $\bar{B}_{*,*}A$ is a complex whose cohomology, i.e. the cohomology of Hom $(\bar{B}_{*,*}A, Z/p)$, is $\mathrm{Ext}_A^{**}(Z/p, Z/p)$ [M1].

The large size of $\bar{B}_{*,*}(A)$ makes direct computation of $\mathrm{Ext}_A^{*,*}(Z/p, Z/p)$ very difficult. If A is a polynomial or exterior algebra, then Koszul constructed a small chain equivalent subcomplex of $\bar{B}_{*,*}(A)$ making computations trivial. Priddy [P1] generalized this construction to an important class of algebras, the homogeneous Koszul algebras. These algebras have generators $\{a_i\}$ and basic relations of the form

$$a_r a_s = \sum c(i,j,r,s)\, a_i a_j \,. \tag{1.4}$$

Note that the length of a word in such an algebra is well defined and gives a secondary grading to A.

It is clear that $\mathcal{R}_p$ is a homogeneous Koszul algebra with generators $\{Q^0, \beta Q^1, Q^1, \ldots\}$ and with the Adem relations (1.2). Since $P^o = 1$, $\mathcal{A}_p$ is not a homogeneous Koszul algebra. For example, $Sq^3 Sq^4 = Sq^7 + Sq^6 Sq^1$ is not a homogeneous relation. The algebra $\mathcal{A}_p(L)$ of Steenrod operations for restricted Lie algebras is isomorphic to $\mathcal{A}_p$ except that the relation $P^o = 1$

is replaced by the relation $P^0 = 0$. It is a homogeneous Koszul algebra. See [P1] and [P2] and [M3] for more details.

Note that A_p and $A_p(L)$ are isomorphic as graded Z/p modules. A basis for each consists of the Steenrod admissible sequences. In the next section we will distinguish between the elements $P^I \in A_p$ and $P_L^I \in A_p(L)$.

DEFINITION 1.5. Let A be a homogeneous Koszul algebra with generators $\{a_i\}$. Then the Koszul subcomplex $\bar{K}_{*,*}(A)$ of $\bar{B}_{*,*}(A)$ is generated by sums of the form $\omega = \sum c_{\alpha,k}\, a_{i_1,\alpha} \otimes \ldots \otimes a_{i_k,\alpha}$

satisfying $d_j\, \omega = 0$ for $j = 1, \ldots, k-1$. The differential on $\bar{K}_{*,*}(A)$ is trivial.

THEOREM 1.6. The homology of $\bar{B}_{*,*}(A)$ is isomorphic to $\bar{K}_{*,*}(A)$.

Proof: See Theorem 3.8 [P1].

EXAMPLE 1.7. If $A = R_p$, then $\bar{K}_{*,1}(A)$ has basis $\{\beta^\varepsilon Q^r\}$. $\bar{K}_{*,2}(A)$ is generated by the "Adem relations"

$$\beta^\varepsilon Q^r \otimes \beta^\delta Q^s - \sum c_t \beta^\eta Q^{r+s-t} \otimes \beta^\xi Q^t.$$

For any k and r, the element $\beta Q^{p^k r} \otimes \ldots \otimes \beta Q^r$ is in $\bar{K}_{*,k}(A)$.

Although Theorem 1.6 theoretically determines the homology and thus the cohomology of R_p, the form of the answer is too complicated. Priddy observed that the dual is far simpler.

DEFINITION 1.8. Let A be a homogeneous Koszul algebra with generators $\{a_i\}$ of degree d_i and relations (1.4). Then the coKoszul complex is the homogeneous Koszul algebra with generators $\{\alpha_i\}$ of degree $d_i + 1$ and relations

$$(-1)^{\nu_{i,j}} \alpha_i \alpha_j = -\sum (-1)^{\nu_{r,s}} c(i,j,r,s) \alpha_r \alpha_s$$

where $\nu_{u,v} = \deg \alpha_u + (\deg(\alpha_u)-1)(\deg(\alpha_v)-1)$.

PROPOSITION 1.9. With the above notation $\mathrm{Ext}_A^{*,*}(Z/p,\ Z/p) \cong \bar{K}^{*,*}(A)$.

Proof: Theorem 2.5 of [P1].

THEOREM 1.10. $\mathrm{Ext}_{R_p}^{*,*}(Z/p,\ Z/p) \cong A_p^{*,*}(L)$.

Proof: The generators of the homogeneous Koszul algebra R_p are the elements $\{\beta^\varepsilon Q^i : \varepsilon = 0$ or 1 and $\varepsilon + i > 0\}$. Thus the generators of its cohomology are corresponding classes $\sigma^{\varepsilon,i}$ of degree $2i(p-1)-\varepsilon+1$. Note that the degree is the same as that of $\beta^\delta P^i \in A_p(L)$ where $\varepsilon + \delta = 1$.

By the equations in (1.2) and (1.8) there are four sets of relations involving $\sigma^{\varepsilon,i}\sigma^{\delta,j}$ depending on the values of ε and δ. For example if $\varepsilon = \delta = 1$, then

$$-\sigma^{1,i}\sigma^{1,j} = -\sum (-1)^{r+j}\binom{(p-1)(j-s)-1}{pj-r-1}\sigma^{1,r}\sigma^{1,s}$$

where the sum is taken over pairs (r,s) with $i+j = r+s$ and $r > ps$. In particular there is no relation for $\sigma^{1,i}\sigma^{1,j}$ if $i \geq pj$ since there are no such terms on the right side of equation (1.2). If we identify $\sigma^{1,i}$ with P^i and use the equality $\binom{u+v}{u} = \binom{u+v}{v}$ we obtain the Adem relation $P^iP^j = \sum (-1)^{i+s}\binom{(p-1)(j-s)-1}{i-ps} P^{i+j-s}P^s$ for $i < pj$.

Similarly if $\varepsilon = 1$ and $\delta = 0$, then the relation becomes

$$\sigma^{1,i}\sigma^{o,j} = -\sum (-1)^{r+j}\binom{(p-1)(j-s)-1}{pj-r}\sigma^{1,r}\sigma^{o,s}$$

$$+\sum (-1)^{r+j}\binom{(p-1)(j-s)}{pj-r}\sigma^{o,r}\sigma^{1,s}$$

corresponding to the Adem relation

$$P^i\beta P^j = -\sum (-1)^{i+s}\binom{(p-1)(j-s)-1}{i-ps-1} P^{i+j-s}\beta P^s$$

$$+\sum (-1)^{i+s}\binom{(p-1)(j-s)}{i-ps}\beta P^{i+j-s}P^s .$$

If $\varepsilon = 0$, then we get similar relations which correspond to applying β to the above (Steenrod) Adem relations.

REMARK 1.11. These results imply that there is a nonsingular pairing

$$A_p^{**}(L)\otimes\bar{K}_{**}(R_p) \to Z/p$$

determined by

$$<\beta^{\varepsilon}P^a,\ \beta^{\delta}Q^b> = \begin{cases} 1 \text{ if } \delta+\varepsilon = 1 \text{ and } a = b \\ 0 \text{ otherwise} \end{cases}$$

and

$$<\beta^{\varepsilon}P^aP^I,\ \sum \beta^{\delta}Q^b\otimes\omega> = \sum <\beta^{\varepsilon}P^a,\ \beta^{\delta}Q^b><P^I,\ \omega>.$$

In particular, $<\beta P^I, \omega> = <P^I, \beta\omega>$.

If $p = 2$ and we use standard notation, then the pairing is determined by

$$<Sq^{a+1},\ Q^b> = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

For example (with standard notation) $Q^8Q^2 + Q^6Q^4 + Q^5Q^5 = 0$. Thus $\omega = Q^8 \otimes Q^2 + Q^6 \otimes Q^4 + Q^5 \otimes Q^5 \in \bar{K}_{10,2}(R_2)$ and $< Sq^9Sq^3, \omega > = <Sq^7Sq^5, \omega > = <Sq^6Sq^6, \omega > = 1$. Note that by the Steenrod Adem relations $Sq^7Sq^5 = Sq^9Sq^3$ and $Sq^6Sq^6 = Sq^{11}Sq^1 + Sq^{10}Sq^2 + Sq^9Sq^3$. This process allows us to compute Adem relations "backwards." For example, to find out which nonadmissible operations Sq^aSq^b contain Sq^9Sq^3 in their admissible expansion, first expand Q^8Q^2 to get $Q^6Q^4 + Q^5Q^5$ and then replace these summands by Sq^7Sq^5 and Sq^6Sq^6.

§ 2. The right action of A_p on $H^*(X; Z/p)$ induces an adjoint action of A_p^{op} on $H_*(X; Z/p)$ determined by $<\beta^\varepsilon P^a u, x> = <u, P^a_* \beta^\varepsilon x>$. If X is an infinite loop space, then the following Nishida relations hold [N], [M4]:

$$\begin{cases} P^a_*Q^c x = \sum (-1)^{a+i} \binom{(c-a)(p-1)}{a-pi} Q^{c-a+i}P^i_* x \\ P^a_*\beta Q^c x = \sum (-1)^{a+i} \binom{(c-a)(p-1)-1}{a-pi} \beta Q^{c-a+i}P^i_* x \\ \qquad + \sum (-1)^{a+i} \binom{(c-a)(p-1)}{a-pi-1} Q^{c-a+i}\beta P^i_* x. \end{cases} \tag{2.1}$$

<u>REMARK 2.2.</u> For p=2 if we identify $\beta^\varepsilon P^a$ with $Sq^{2a+\varepsilon}$ and, as in section 1, $\beta^\delta Q^c$ with $Q^{2c-\delta}$, then the standard Nishida relations follow:

$$Sq^a_* Q^c = \sum \binom{c-a}{a-2i} Q^{c-a+i} Sq^i_* .$$

These formulae are not sufficient to make R_p into an A_p^{op} module. For example with p=2 and using standard notation $Q^5Q^0 = Q^1Q^4$ but $Sq^2{}_*(Q^5Q^0) = Q^3Q^0 = Q^1Q^2 \neq 0 = Sq^2{}_*(Q^1Q^4)$. It is probably possible to extend the Nishida relations so that R_p does become an A_p^{op} module. Indeed there is evidence to believe that this will happen if we make the conventions that $Q^i = 0$ if $i < 0$ and $\binom{-m}{-n} = \binom{m}{n}$ if $m, n \geq 0$. Rather than pursue this avenue, we opt for the following simpler and geometrically more natural approach.

Recall [CLM] that if $I = (\varepsilon, a_1, \ldots, \varepsilon_k, a_k)$ is a sequence such that $\varepsilon_i = 0$ or 1 and $a_i \geq \varepsilon_i$, then the (Dyer Lashof) excess of I (or Q^I) is

$$e(I) = 2a_1 - \varepsilon_1 - \sum_{j=2}^{k} [2a_j (p-1)-\varepsilon_j]. \tag{2.3}$$

Let E(0) be the ideal of R_p generated by the Q^I of negative excess. Then the algebra $R_p(0) = R_p/E(0)$ is isomorphic to $QH_*(\Omega^\infty S^\infty; Z/p)$, the quotient module of indecomposable homology classes of $\Omega^\infty S^\infty = \lim_{\to} \Omega^N S^N$ [DL], [M4]. It follows immediately from this isomorphism that $R_p(0)$ is an A_p^{op} module.

Now let $\bar{B}_{*,k}(0)$ be the subcomplex of $\bar{B}_{*,k}(R_p)$ generated by elements $Q^{I_1} \otimes \cdots \otimes Q^{I_k}$ satisfying $(I_1,\ldots,I_k) = (J,K,L)$ where $e(K) < 0$. If we assume that I_j is (Dyer Lashof) admissible for each j, then this condition is equivalent to saying that for some j, $e(I_j) < \sum_{t>j} \deg(Q^{I_t})$. Alternatively $Q^{I_1} \ldots Q^{I_k}$ vanishes on all homology classes purely for excess reasons.

In analogy with Miller's definition of UnTor [M5], we define $\mathrm{UnExt}_{R_p}^{*,*}(Z/p, Z/p)$ to be the cohomology of the Z/p dual of $\bar{B}_{*,*}(R_p)/\bar{B}_{*,*}(0)$.

PROPOSITION 2.4. $\mathrm{UnExt}_{R_p}^{*,k}(Z/p, Z/p) \approx \bar{K}^{*,k}(R_p) \approx A_p(L)^{*,k}$ as Z/p modules.

Proof: As noted by Miller in Proposition 3.1.2 [M5], Priddy's proof of our Theorem 1.6 extends to show that $\bar{B}_{*,*}(R_p)/\bar{B}_{*,*}(0)$ is chain equivalent to $\bar{K}_{*,*}(R_p)/\bar{K}_{*,*}(R_p) \cap \bar{B}_{*,*}(0)$. Since (Dyer Lashof) Adem relations involve elements Q^aQ^b with $a > pb$, in particular of positive excess, it is easy to see that elements of $\bar{K}_{*,k}(R_p)$ are sums with at least one summand not in $\bar{B}_{*,*}(0)$. Thus $\bar{K}_{*,*}(R_p)/(\bar{K}_{*,*}(R_p) \cap \bar{B}_{*,*}(0))$ is isomorphic to $\bar{K}_{*,*}(R_p)$.

This proposition says that $A_p(L)$ may be considered to be either the module generated by all sequences $\beta^{\varepsilon_1} P^{a_1} \ldots \beta^{\varepsilon_k} P^{a_k}$ subject to the Adem relations or by the subset of those sequences satisfying $2a_j + \varepsilon_j \geq \Sigma \deg \beta^{\varepsilon_t} P^{a_t}$ subject to the same relations. Since each module has the set of admissible sequences as basis, this is obvious.

REMARK 2.5. Consider the quotient complex $\bar{B}_{*,*}(R_p)/\bar{B}_{*,*}(m)$ "generated" by $Q^{I_1} \otimes \cdots \otimes Q^{I_k}$ with I_j admissible and with $e(I_j) \geq (\sum_{t>j} \deg Q^{I_t}) + m$ for all j. Let T_*^m be the graded module $T_n^m = Z/p$ if $m = n$ and 0 if $m \neq n$. Then, following Miller, we can define $\mathrm{UnExt}_{R_p}(Z/p, T_*^m)$ to be the cohomology of $(\bar{B}_{*,*}(R_p)/\bar{B}_{*,*}(m))^*$. It can be shown that this cohomology is isomorphic to the submodule of $A_p(L)$ generated by admissible sequences $\beta^{\varepsilon_1} P^{a_1} \ldots \beta^{\varepsilon_k} P^{a_k}$ with $2a_k + \varepsilon_k > m$. This construction is important in studying the Miller spectral sequence [M5], [KL1], [KL2].

The (Steenrod) Adem relations induce a right A_p module structure on $A_p(L)^{*,k}$ as follows. Assume that $\beta_L^{\varepsilon} P_L^a P_L^I \in A_p(L)^{*,k}$ is admissible where $\varepsilon = 0$ or 1 and we distinguish elements of $A_p(L)$ by using the subscript L. Then $\beta(\beta_L^{\varepsilon} P_L^a P_L^I) = (\varepsilon + 1)\, \beta_L P_L^a P_L^I$, $P^a P_L^I = 0$ if $a \neq 0$ and $I = \emptyset$, and

$$P^a P_L^b P_L^I = \begin{cases} \sum (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P_L^{a+b-i} P^i P_L^I & \text{if } a < pb \\ 0 & \text{if } a \geq pb \end{cases}$$

$$P^a \beta_L P_L^b P_L^I = \begin{cases} \sum (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta_L P_L^{a+b-i} P^i P_L^I & \text{if } a \leq pb \\ + \sum (-1)^{a+i-1} \binom{(p-1)(b-i)-1}{a-pi-1} P_L^{a+b-i} \beta_L P^i P_L^I & \\ 0 & \text{if } a > pb. \end{cases} \qquad (2.6)$$

REMARK 2.7. The assumption that $P_L^b P_L^I$ be admissible is necessary. For example, $Sq^5 Sq^2{}_L = 0$ while $Sq^5(Sq^2{}_L Sq^5{}_L) = Sq^5(Sq^6{}_L Sq^1{}_L) = Sq^{11}{}_L Sq^1{}_L$.

THEOREM 2.8. $\mathrm{Ext}_{R_p}^{*,*}(Z/p, Z/p) \approx A_p(L)^{*,*}$ as bigraded A_p modules.

Proof: We must show that

$< P^a P_L^J, \omega > = < P_L^J, P_*^a \omega >$ where $P_L^J = \beta^{\varepsilon} P_L^b z$ is admissible in $A_p(L)$ and where $\omega \in \bar{K}_{*,k}(R_p)$. Assume inductively that this equation holds for sequences of length $< k$. Write $\omega = \sum_{\delta, c} \beta^{\delta} Q^c \otimes \omega'$ for $\omega' = \omega'_{\delta,c} \in \bar{K}_{*,k-1}(R_p)$.

For p=2, this theorem is essentially proven in Section 4 of [M5]. The general proof breaks up into four cases depending on ε and δ.

CASE 1. $\varepsilon = \delta = 0$. Then by (1.11)

$$< P^a P^b_L z, Q^c \otimes \omega' > = \sum \gamma_t < P_L^{a+b-t} P^t z, Q^c \otimes \omega' >$$

$$= 0$$

and

$$< P^b_L z, P^a_* (Q^c \otimes \omega') > = \sum \eta_t < P^b_L z, Q^{c-a+t} \otimes P^t_* \omega' >$$

$$= 0$$

CASE 2. $\varepsilon = 0$ and $\delta = 1$.

We first expand the left side to obtain

$$< P^b_L z, P^a_* \beta Q^c \otimes \omega > = \sum (-1)^{a+i} \binom{(c-a)(p-1)-1}{a-pi} < P^b_L z, \beta Q^{c-a+i} \otimes P^i_* \omega' >$$

$$+ \sum \gamma_i < P^b_L z, Q^{c-a+i} \otimes \beta P^i_* \omega' >$$

$$= \begin{cases} (-1)^{a+i} \binom{(c-a)(p-1)-1}{a-pi} < z, P^i_* \omega' > & \text{if } b = c-a+i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this is 0 if $(c-a)(p-1)-1 < a-pi$, i.e. if $b = c-a+i \le \frac{c}{p}$.

If $a \geq pb$ so that $P^a P^b_L z = 0$ by (2.6), then either $a > c$ so that $P^a_* \beta Q^c = 0$, or $c \geq a \geq pb$ so that the binomial coefficient above is 0. If $a < pb$, then

$$< P^a P^b_L z, \beta Q^c \otimes \omega' > = \sum (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} < P_L^{a+b-i} P^i z, \beta Q^c \otimes \omega' >$$

$$= \begin{cases} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} < P^i z, \omega' > & \text{if } a+b-i = c \\ 0 & \text{if } a+b-i \neq c. \end{cases}$$

Thus the two coefficients agree and $< P^i z, \omega' > = < z, P^i_* \omega' >$ by the induction hypothesis.

The cases where $\varepsilon = 1$ are almost identical and will be left to the reader.

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